

## 2003 AB1/BC1

1a. The point of intersection of  $y = \sqrt{x}$  and  $y = e^{-3x}$  is  $x = 0.238734$  so the

integral for the area is  $\int_{0.238734}^1 (\sqrt{x} - e^{-3x}) dx = 0.44263$

1b. The volume is found by using washers

$$\pi \int_{0.238734}^1 \left( (1 - e^{-3x})^2 - (1 - \sqrt{x})^2 \right) dx = 1.42356$$

1c. The base of the rectangle is  $(\sqrt{x} - e^{-3x})$ . The height of the rectangle is

5. So the volume is  $\int_{.238734}^1 5(\sqrt{x} - e^{-3x})^2 dx = 1.55435$

## 2003 AB2

2a. Acceleration is the derivative of velocity so

$$\left. \frac{dy}{dt} = \frac{d\left(- (t+1) \sin\left(\frac{t^2}{2}\right)\right)}{dt} \right|_{t=2} = 1.58758$$

$v(2) < 0$ . So since acceleration is positive at  $t = 2$  when  $v(2) < 0$ . The speed is decreasing at  $t = 2$ .

2b. Particle changes direction when  $v(t) = 0$ . Solving this graphically gives  $t = 2.50663$ . A sign study of  $v(t)$  indicates that  $v(t)$  is negative from  $(0, 2.50663)$  and positive from  $(2.50663, 3)$ .

2c. Total distance

$$\int_0^{2.50663} -v(t) dt + \int_{2.50663}^3 v(t) dt = 4.33382$$

2d. . This is the position function. There are three possibilities for the maximum distance from the origin. At the start of the problem when  $t = 0$ , when the particle turns around at  $t = 2.50663$ , or when the problem ends at  $t = 3$ . Find the position at each of these times yields the following:

t	x(t)
0	1
2.20663	-2.26548
3	5.33382

The maximum distance from the origin is 5.33382 at  $t = 3$ .

## 2003 AB3

3a.  $R'(45)$  is approximately equal to  $\frac{R(40) - R(50)}{40 - 50} = 1.5 \frac{\text{gal} / \text{min}}{\text{min}}$

3b. At  $t = 45$  the function  $R(t)$  is increasing fastest. After this point the slope of  $R(t)$  will decrease. This is the location where  $R''(t)$  is changing from positive concavity to negative concavity so  $R''(t)$  should be zero.

3c.

$$\int_0^{90} R(t) dt \approx 30(20) + 30(10) + 40(10) + 55(20) + 65(20)$$
$$\int_0^{90} R(t) dt \approx 3700 \text{ gal}$$

This value is less than the actual value since all rectangles are under the function or  $R(t)$  is increasing for  $0 < t < 90$  so the Left hand Riemann Sum is less than the actual area.

3d.  $\int_0^b R(t) dt =$  is the number of gallons of fuel consumed during the first  $b$  minutes of a flight.

$\frac{1}{b} \int_0^b R(t) dt =$  average rate of consumption (gallons/minute) of fuel consumed during the first  $b$  minutes of the flight.

## 2003 AB4/BC4

4a. A sign study of  $f'$  tells where the function  $f$  is increasing and decreasing

x	-3 < x < -2	-2 < x < 0	0 < x < 2	2 < x < 3
f'(x)	positive	negative	negative	negative

So  $f$  is increasing between  $-3 < x < -2$ .

4b. A sign study of  $f''$  tells where  $f$  concavity.

x	-3 < x < 0	0 < x < 2	2 < x < 3
f''(x)	negative	positive	negative

So  $f$  has a point of inflection at  $x = 2$ . At  $x = 0$  the second derivative does not exist.

4c.  $f'(0) = -2$  so the tangent line is  $y = -2x + 3$ .

4d. Looking at the graph you will notice

that  $\int_{-3}^0 f'(x) dx = f(x) \Big|_{-3}^0 = f(-3) - f(0)$

Then  $-1\frac{1}{2} = f(-3) - f(0)$  based on the

bounded area from  $[-3, 0]$ .

This leads to  $-1\frac{1}{2} = f(-3) - 3$  since  $f(0) = 3$ .

Solving yields  $-4\frac{1}{2} = f(-3)$

Alternately for this first question you can also find the antiderivative of the straight line, solve for  $c$  and then find

$f(-3)$  by substitution in the function.

$$f(x) = -\frac{x^2}{2} - 2x + c$$

$$f(0) = 3 = c$$

$$f(x) = -\frac{x^2}{2} - 2x + 3$$

$$f(-3) = -4\frac{1}{2}$$

To find  $f(4)$  use the same technique with area:

$$\int_0^4 f'(x) dx = f(x) \Big|_0^4 = f(4) - f(0)$$

$$8 - \pi = f(4) - f(0)$$

$$-8 + \pi = f(4) - 3$$

$$f(4) = -5 + \pi$$

5a.

$$V = \pi r^2 h$$

$$V = 25\pi h$$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

you are given the rate of change of V so

$$-5\pi\sqrt{h} = 25\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi}$$

$$\frac{dh}{dt} = \frac{-\sqrt{h}}{5}$$

5b.

$$\frac{dh}{dt} = \frac{-\sqrt{h}}{5}$$

$$\frac{dh}{\sqrt{h}} = \frac{-1}{5} dt$$

$$\int \frac{dh}{\sqrt{h}} = \int \frac{-1}{5} dt$$

$$2\sqrt{h} = \frac{-1}{5}t + C$$

When  $t = 0$ ,  $h = 17$ .

$$2\sqrt{17} = \frac{-1}{5}0 + C$$

$$C = 2\sqrt{17}$$

and

$$2\sqrt{h} = \frac{-1}{5}t + 2\sqrt{17}$$

$$\sqrt{h} = \frac{-1}{10}t + \sqrt{17}$$

$$h = \left( \frac{-1}{10}t + \sqrt{17} \right)^2$$

5c. The coffee pot is empty when  $h = 0$ .

$$h = \left( \frac{-1}{10}t + \sqrt{17} \right)^2$$

$$0 = \left( \frac{-1}{10}t + \sqrt{17} \right)^2$$

$$0 = \frac{-1}{10}t + \sqrt{17}$$

$$t = 10\sqrt{17} \text{ sec}$$

# 2003 AB6

6a.  $\lim_{x \rightarrow 3^+} f(x) = 2 = \lim_{x \rightarrow 3^-} f(x)$  so the function is continuous at  $x = 3$ .

6b. The average value has been calculated by finding the area under the function and dividing by the width.

$$\begin{aligned} f_{avg}(x) &= \frac{1}{5} \left( \int_0^3 f(x) dx + \int_3^5 f(x) dx \right) \\ &= \frac{1}{5} \left( \frac{2}{3} (x+1)^{3/2} \Big|_0^3 + 5x - \frac{x^2}{2} \Big|_3^5 \right) \\ &= \frac{1}{5} \left( \frac{2}{3} \left( (4)^{3/2} - 1 \right) + \left( \left( 25 - \frac{25}{2} \right) - \left( 15 - \frac{9}{2} \right) \right) \right) \\ &= \frac{1}{5} \left( \frac{2}{3} (7) + \left( \frac{25}{2} - \frac{21}{2} \right) \right) \\ &= \frac{1}{5} \left( \frac{14}{3} + 2 \right) \\ &= \frac{1}{5} \left( \frac{20}{3} \right) \\ &= \frac{4}{3} \end{aligned}$$

6c. Following a similar step to part a yields:

$$\lim_{x \rightarrow 3^+} f(x) = 3m + 2 = 2k = \lim_{x \rightarrow 3^-} f(x)$$

$$\text{so } 3m + 2 = 2k \quad (\text{A})$$

Since  $g$  is differentiable then

$$g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}}, & 0 < x < 3 \\ m, & 3 < x < 5 \end{cases}$$

The question is do the two derivative parts equal each other as  $x$  approaches 3.

For this to be true

$$\frac{k}{2\sqrt{3+1}} = m$$

$$\frac{k}{4} = m \quad (\text{B})$$

$$k = 4m$$

Solving the two equations A and B together gives

$$3m + 2 = 2(4m)$$

$$m = 2/5 \quad k = 8/5$$

# 2003 BC2

2a. At point C  $\frac{dy}{dx}$  is positive.

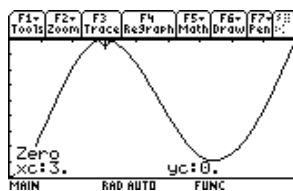
$$\text{Since } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

both  $dy/dt$  and  $dx/dt$  must have the same sign.  $dy/dt$  is negative since the vertical change is in the negative direction.  $dx/dt$  is negative since the particle is moving to the left.

2b.  $dy/dx$  is undefined at b.

$$\text{Since } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$dx/dt$  must be equal to zero. Looking at a graph of  $dx/dy$

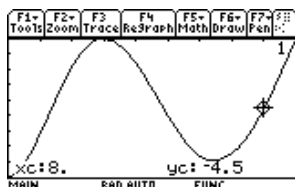


shows us that at  $t = 0$  the  $dx/dt$  is zero.

2c. Since  $y = \frac{5}{9}x - 2$ ,

$$\frac{dy/dt}{dx/dt} = \frac{5k}{9k}$$

Looking at the graph of  $x'(t)$  and specifically at  $x'(8)$



we can see that when  $t = 8$   $dx/dt = -4.5$ .

Therefore  $-4.5 = 9k$  or  $k = -0.5$ . So  $dy/dt = -2.5$ .

The velocity vector is  $\langle -4.5, -2.5 \rangle$

The speed of the particle is

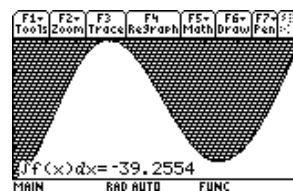
$$\sqrt{(-4.5)^2 + (-2.5)^2} = 5.14782$$

2d.

$$x(t) = \int_0^t -9 \cos\left(\frac{\pi x}{6}\right) \sin\left(\frac{\pi\sqrt{x+1}}{2}\right) dx + x_A$$

$$x(0) = \int_0^0 -9 \cos\left(\frac{\pi x}{6}\right) \sin\left(\frac{\pi\sqrt{x+1}}{2}\right) dx + x_A$$

$$= x_A$$



$$x(0) = \int_0^9 -9 \cos\left(\frac{\pi x}{6}\right) \sin\left(\frac{\pi\sqrt{x+1}}{2}\right) dx + x_A$$

$$= -39.2554 + x_A$$

So the particle has moved -39.2554 units to the left.

# 2003 BC3

3a. Solve the two equations together to find the point of intersection.

$$\frac{5}{3}y = \sqrt{1+y^2}$$

$$\frac{25}{9}y^2 = 1+y^2$$

$$\frac{16}{9}y^2 = 1$$

$$y^2 = \frac{9}{16}$$

$$y = \frac{3}{4}$$

$$x = \frac{5}{3}$$

$$\left(\frac{5}{3}, \frac{3}{4}\right)$$

$$3b. = .3 \int_0^{\frac{3}{4}} \left( \sqrt{1+y^2} - \frac{5}{3}y \right) dy = 0.346574$$

3c.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 - y^2 = 1$$

$$(r \cos \theta)^2 - (r \sin \theta)^2 = 1$$

$$r^2(\cos^2 \theta - \sin^2 \theta) = 1$$

$$r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}$$

3d.

$$\frac{5}{3} = r \cos \theta$$

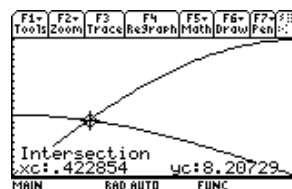
$$\frac{3}{4} = r \sin \theta$$

$$\therefore r = \frac{3}{4 \sin \theta} = \frac{5}{3 \cos \theta}$$

$$\frac{3}{4 \sin \theta} = \frac{5}{3 \cos \theta}$$

$$9 \cos \theta = 20 \sin \theta$$

solving these together yields



or

$$\frac{1}{2} \int_0^{0.422854} r^2 d\theta$$

$$= \frac{1}{2} \int_0^{0.422854} \frac{1}{\cos^2 \theta - \sin^2 \theta} d\theta$$

## 2003 BC6

$$6a. \quad f'(x) = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \dots + \frac{(-1)^n (2nx^{2n-1})}{(2n+1)!}$$

$$f'(0) = 0$$

$$f''(x) = \frac{-2}{3!} + \frac{4 \cdot 3x^2}{5!} - \frac{6 \cdot 5x^4}{7!} + \dots + \frac{(-1)^n (2n \cdot (2n-1)x^{2n-2})}{(2n+1)!}$$

$$f''(0) = -\frac{1}{3}$$

Since  $f''(0) = -\frac{1}{3}$  we can use the second derivative test and determine that since the first derivative is equal to zero at zero and the second derivative is negative the function  $f$  has a maximum at  $x = 0$ .

$f'(0) = 0$  so let's use the first derivative test.

When  $x < 0$   $f'(x) < 0$ . When  $x > 0$   $f'(x) > 0$ . So since the sign of  $f'$  changes from negative to positive there must be a local maximum at  $x = 0$ .

6b.  $f(1) \approx 1 - \frac{1}{3!}$  for a two term approximation of  $f(1)$ . So the most error will be the next

term.  $\frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$

6c

$$f'(x) = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \dots + \frac{(-1)^n (2nx^{2n-1})}{(2n+1)!}$$

$$xf'(x) = \frac{-2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \dots + \frac{(-1)^n (2nx^{2n})}{(2n+1)!}$$

$$f'(x) = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \dots + \frac{(-1)^n (2nx^{2n-1})}{(2n+1)!}$$

$$xf'(x) + f(x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots + \frac{(-1)^n (x^{2n})}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n (x^{2n})}{(2n)!}$$

$$= \cos x$$