2007 BC6

A.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...$$

So

$$e^{-x^2} = 1 + \left(-x^2\right) + \frac{\left(-x^2\right)^2}{2!} + \frac{\left(-x^2\right)^3}{3!}$$

$$= 1 - \left(x^2\right) + \frac{\left(x^4\right)}{2!} - \frac{\left(x^6\right)}{3!}$$

B.

$$\lim_{x\to 0} \frac{1 - x^2 - \left(1 - \left(x^2\right) + \frac{\left(x^2\right)^2}{2!} - \frac{\left(x^2\right)^3}{3!}\right)}{x^4}$$

$$= \lim_{x\to 0} \frac{\left(-\frac{x^4}{2} + \frac{x^6}{6}\right)}{x^4}$$

$$= \lim_{x\to 0} \left(-\frac{1}{2} + \frac{x^2}{6}\right)$$

$$= -\frac{1}{2}$$

C.

$$\int_0^x e^{-t^2} dt = \int_0^x \left(1 - \left(t^2\right) + \frac{\left(t^2\right)^2}{2!} - \frac{\left(t^2\right)^3}{3!} + ...\right) dt$$

$$= t - \frac{t^3}{3} + \frac{t^5}{10} - \frac{t^7}{42} + ...\Big|_0^x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + ...$$

$$\int_0^{1/2} e^{-t^2} dt \approx \left(1/2\right) - \frac{\left(1/2\right)^3}{3}$$

$$or\ 1/2 - 1/24\ or\ \frac{11}{24}$$

D. The estimate found in part c differs from the actual value of $\int_0^{-1/2} e^{-t^2} dt$ by less than 1/200 because when the integral is estimated with a certain number of terms of the Taylor Series the error is less than the absolute value of the next term or in this case $\frac{\left(\frac{1}{2}\right)^5}{10} = \frac{1}{3200}$. And we know that 1/320 is less than 1/200. So the error is less than 1/200.