Introducing Series and Convergence

Polynomials can be built that resemble a function in a small interval.

Recall that if a line T(x) is drawn tangent to a function f(x) at a point x = a two properties are true

a. T(a) = f(a) (there is a point in common)
b. T'(a) = f'(a) (the slope is the same)

These ideas can be translated to property of local linearity: A differentiable function, over a small interval, resembles a straight line. This means that we can approximate values on a function at a point near x = a, but finding values along a tangent line.

Suppose $f(x) = x^2$. Write a equation for the tangent line at x = 2.

T(x) =

Use your graphing calculator to graph a picture of f(x) and T(x) in a small window near x=2.

Create a set of table values near x = 2 with ΔtI = 0.1

What do you notice from the graph and the table?

The straight lines does not perfectly match up with the parabola, but gives us values very close to those on the function when we are near the point of tangency. Suppose we think of a different function such as y=ln x.

Again find the tangent line to y=ln x at x = 2 and graph the tangent line along with it's tangent line at x = 2.

T(x) =

Would a parabola fit the curve better than the straight line at x = 2? Recall that the tangent line was of the form $T(x) = f'(2)(x-2) + f(2)$ or, in general, $T(x) = f'(a)(x-a) + f(a)$ A general parabola that could be: $y = c_2(x-a)^2 + c_1(x-a) + f(a)$. Notice that at x=a, the value on the parabola would equal f(a).

Let's solve for c_1 and c_2. This parabola and function f(x) should have the same tangent line at x=a so the derivative of f and the first derivative of the parabola should be equal.

so at x = a

c_1 =

Since the parabola and the functions should curve the same way their concavity should be the same so

c_2 =

Therefore, the parabola that fits the function f(x) would have the form

P(x) =____

Let's find the parabola that fits the function y=ln x.

P(x) = _______________________________________

Graph the function, it's tangent line and it's parabola at x=2. Look at a set of table values for the three functions near x = 2.

What do the graph and the table values illustrate?

Continuing this process, the cubic polynomial that fits the function f(x) would be

$$C(x) = c_3(x-a)^3 + c_2(x-a)^2 + c_1(x-a) + f(a)$$

Solve for the values of c_1, c_2, and c_3, using similar reason that we used with the parabola.

Study patterns that you see developing to write the polynomial of degree four that fits the function.

P(x) =

These equations are called Taylor polynomials or various degrees. Enter each new polynomial in the calculator and view the graphs and tables associated with each. Each time you enter a new polynomial notice you can start with the previous equation and add only one new term. This will make it easier to enter the new equation.

What do you notice about the Taylor polynomials?

1. As the degree of the polynomial increases the graph of the new polynomial comes closer to the graph of the function than the previous polynomial.

2. The polynomials exist outside the domain of the function f.

3. The interval where the graphs are close to the function f is limited.

Write out a general Taylor polynomial for a function f.

What would happen if we continued to add more terms to the Taylor polynomial to create a series?

What would happen if we formed a series with an infinite number of terms? We call this a Taylor series.

Exercises: For each function, graph the function near x = a and write several Taylor Polynomials. View the graphs and table values for each.

1. $y = e^x$ near x = 0

2. y= cos x near x = 0

Let's study y = sin x near x = 0

Write out the Taylor polynomials of degree 1, 3, 5, 7, and 9.

$T_1(x) =$

$T_3(x) =$

$T_5(x) =$

$T_7(x) =$

$T_9(x) =$

Graph each Taylor polynomial against the original function.

Did you notice that as you added more terms on the Taylor polynomial for y = sin x the interval where the curves matched up seemed to get wider.

Write out the Taylor series for y=sin x.

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Notice as you add terms to the Taylor polynomial the Taylor polynomial approaches the function y = sin x.

Set up a new equation that compares the Taylor polynomial with the original function: $y = \left| y1 - T(x) \right|$. Set up a window : $-4 \le x \le 4$ and $-0.2 \le y \le 0.2$. If $T(x)=T_3(x)$ then this equation is describing the error between the function y = sin x and the $T_3(x) = -\frac{1}{3!}(x-0)^3 + 1(x-0) + 0$. Keep changing the Taylor polynomial and notice what happens to this graph.

The interval where the Taylor series comes close to the function is called an interval of convergence. We say that the series converges when the x values are within the interval. When the x values are outside the interval the series diverges. Interval of convergence are centered around the point x = a and may be opened, closed, or half open.

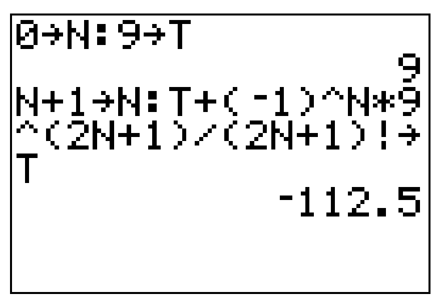
Describe what this graph is telling you about the two functions.

Remember the Taylor polynomial for y= sin x is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. How many terms of the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ are required to approximate sin 9 accurately to the third decimal place?

First find the sin 9 on your calculator:

sin 9 = .4121184852

Now let's set up a recursive set of steps that calculated one additional term every time we press enter.



Line 1: 0 sto N: 9 sto T Press ENTER
Line 2: N+1 sto N: T+(-1)^N*9^(2N+1)/(2N+1)! Press ENTER

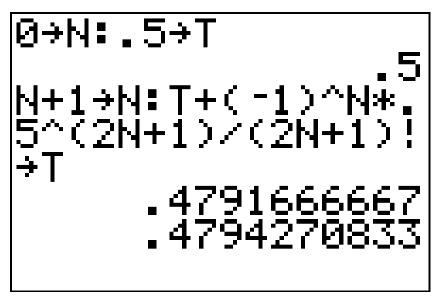
The first time you press enter you have the value of third degree Taylor polynomial at x = 9. This is a value along a cubic equation that closely fits y=sin x near x=0.

Each time you press ENTER you are adding one more term to the polynomial. Keep track of the degree of the polynomial, the value, and stop when you have accuracy to three decimal places.

Degree of Taylor Polynomial	Approximate value at x = 9 along the polynomial

What happens when we pick a value of x nearer to x=0 such as x = 0.5

Rewrite the two lines to fit x = 0.5. Find the value for sin 0.5



sin .5 = .479426

Line 1: 0 sto N: .5 sto T Press ENTER
Line 2: N+1 sto N: T+(-1)^N*.5^(2N+1)/(2N+1)! Press ENTER

The first time you press enter you have the value of third degree Taylor polynomial at x = 0.5. This is a value along a cubic equation that closely fits y=sin x near x=0. What do you notice.