Strategy for Solving Max-Min Problems

The Problem
Designing a Can

A can shaped like a right cylinder (including both top and bottom) with volume 1 cubic foot is to be made from material costing $0.80 per square foot. A vertical seam and the top and bottom are soldered at a cost of $0.20 per linear foot. Find the dimensions of the can of minimum cost and the minimum cost graphically, and confirm analytically.

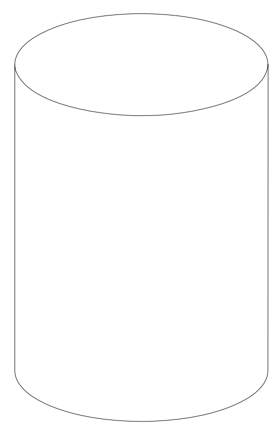
1. Understand the Problem

Read the problem carefully. Identify the information you need to solve the problem.

The formula for the volume of the cylinder and the cost to produce the can will be needed. The cost equation is based on two part: the surface area and the length of the height.

$$Volume = \pi r^2 h \qquad Cost = .80\left(2\pi r^2 + 2\pi rh\right) + .20(h) + .20(2)(2\pi r)$$

2. Develop a Mathematical Model of the Problem



Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.

Volume = 1 cubic foot

$$1 = \pi r^2 h \qquad C = .80\left(2\pi r^2 + 2\pi rh\right) + .20(h) + (.80\pi r)$$

By inspection we can use the first equation to replace either r or h in the second equation. Since h appears a one factor in two terms in the cost equation let's solve for h.

$\dfrac{1}{\pi r^2} = h$. Replace h in the second equation yields

$$C = \left(1.6\pi r^2 + 1.6\pi r\left(\frac{1}{\pi r^2}\right)\right) + .20\left(\frac{1}{\pi r^2}\right) + (.80\pi r)$$

$$C = 1.6\pi r^2 + \left(\frac{1.6}{r}\right) + \left(\frac{.20}{\pi r^2}\right) + (.80\pi r)$$

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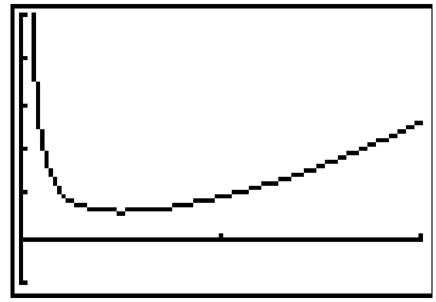
This equation describes the Surface Area of the can as a function of just the height.

3. Graph the Function



The graph of the cost for $0 \le r \le 2$

Find the domain of the function. Determine what values of the variable make sense in the problem.

$$C = 1.6\pi r^2 + \left(\frac{1.6}{r}\right) + \left(\frac{.20}{\pi r^2}\right) + (.80\pi r)$$

From this equation we can see that $r > 0$ or r is an element of $(0,\infty)$. As r increases to large numbers (and the volume remains 1 cubic foot) the cost of producing the can will simply be the area of 2 large circles each with a radius r times $0.80 and .20 times two circumferences.

We can notice from the graph that as x begins near zero the cost of the can is very large. (Caused by the second term in the cost formula.) Then as x increases the cost decreases to a minimum value at about x = 0.5 and then begins to increase again.

4. IdentifY the Critical Points and Endpoints

Find where the derivative is zero or fails to exist.

$$C' = 3.2\pi r - \frac{1.6}{r^2} - \frac{.40}{\pi r^3} + .80\pi$$

$$C' = \frac{3.2\pi^2 r^4 - 1.6\pi r - .40 + .80\pi^2 r^3}{\pi r^3}$$

Since r must be greater than zero we can ignore that the derivative does not exist at r = 0.

$$C' = \frac{3.2\pi^2 r^4 - 1.6\pi r - .40 + .80\pi^2 r^3}{\pi r^3}$$

Therefore the only critical value is where the numerator equals zero. Using the calculator the zero between $0 \le r \le 2$ is at r = ..497118.

There are no endpoint values in this problem since the domain is $(0,\infty)$

5. Solve the Mathematical Model

If unsure of the result, support or confirm your solution with another method.

C' changes from being negative to positive at r = .497118 This means that cost is first decreasing and then increasing. Therefore this value of r is the location of a local minimum. Since there is only one sign change in the derivative the cost continues to grow as r increases past r= .497118

6. Interpret the Solution.

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Translate your mathematical result into the problem setting and decide whether the result makes sense. State your solution in a complete English sentence. Be sure to state all units.

The cost of producing the can is when the radius is .497118 ft.

A second problem to consider the same way:

Jane is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time.

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