Area of a Circle
An Opportunity for Students Experience Limits Visually and Numerically

Both the NCTM Curriculum and Evaluation Standards for School Mathematics and the New Jersey Curriculum Framework suggestion that students be introduced to the underpinnings of calculus in each grade cluster: K-4, 5-8, and 9-12. Standard 13 in the 9-12 cluster of the Curriculum and Evaluation Standards states that students should be given experience studying both graphically and numerically the idea of a limit. Standard 15 in the 9-12 cluster of the New Jersey Mathematics Curriculum Frameworks calls for students to have the opportunity to informally investigate the central ideas of calculus such as the limit, the rate of change, the area under a curve, and the slope of a tangent line.

Developing a formula for the area of circle can be investigated in a high school geometry class through an activity which encourages some of these ideas suggested by these two standards. First students need to determine a way to find the area of a regular polygon. This can be accomplished by giving students regular polygons with different number of sides and developing a pattern for the area of a 3, 4, 5, 6, ..., n-sided polygon.

NUMBER OF SIDES	3	4	5	6	. . .	n
NUMBER OF CONGRUENT TRIANGLES	3	4	5	6		n
AREA OF ONE TRIANGLE	$\frac{1}{2}(as)$	$\frac{1}{2}(as)$	$\frac{1}{2}(as)$	$\frac{1}{2}(as)$		$\frac{1}{2}(as)$
TOTAL AREA	$\frac{1}{2}(as)(3)$	$\frac{1}{2}(as)(4)$	$\frac{1}{2}(as)(5)$	$\frac{1}{2}(as)(6)$		$\frac{1}{2}(as)(n)$

From the last row of the chart students can see that all these expressions can be re-written to be $\frac{1}{2}(a)(ns)$. Since *ns = Perimeter* the area formula can then be re-written as $\frac{1}{2}(aP)$.

This formula can be used to find the area of any regular polygon with an apothem *a* and a perimeter *P*. Students can then study additional regular polygons to see what patterns develop as the number of sides increases.

I gave my students a copy of a two part activity sheet. One contained several regular polygons and a circle. All the figures had one quality in common: all of the regular polygons could be inscribed in the circle. But the students did not know the common characteristic when they started to measure the figures. The other part gave the directions on what to measure and what areas to calculate. Students were instructed to measure and find the following quantities: number of sides, length of the radius of circumscribed circle, length of the apothem, length of the one side of the polygon, perimeter of the regular polygon, area of the representative triangle, and area of the polygon.

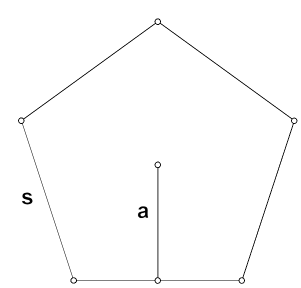

Figure 1

With students working in groups it was possible for each group to collect the data in about 20 minutes. Then students were asked to make observations about the chart. They were asked to study the chart and describe how the various measurements were changing as the number of sides of the polygon increased.

Students made the following observations:

As n increased

- the radius of the polygon/circle remained constant
- the length of the apothem decreased
- the length of the perimeter of the regular polygon increased
- the length of the base of each triangle decreased
- the area of each triangle decreased
- the number of congruent triangles increased
- the area of the regular polygon increased

Since the goal of the lesson was to find an equation which could be used to find the area of a circle I also asked the students to think about the maximum value or minimum value each of these quantities might be approaching. This was done by thinking about the restraints the circumscribed circle placed on the regular polygon. The ideas which came out of the class discussion included:

- What value was the apothem approaching? As the number of sides of the polygon increased the apothem was getting closer and closer to the radius of the circle.
- What value was the perimeter approaching? As the number of sides of the polygon increased the perimeter of the polygon as getting closer and closer to the perimeter of the circle.
- What length does the base of the triangle approach? As the number of sides of the polygon increased the length of the base decreases to zero.
- What value was the area of the triangle approaching? As the number of sides of the polygon increased the area of each triangle was decreasing and would approach zero.
- What value was the number of congruent triangles approaching? There was not number this was approaching. The students decided that the number of triangles could continue to increase. There is not limit on the number of triangles that can fit in the circle. A polygon can have a very large number of sides.
- And finally, what value was the area of the regular polygon approaching? As the number of sides of the polygon increased the area of the regular polygon approached $\frac{1}{2}\left(r\left(2\pi r\right)\right)$ or πr^2.

It would also be possible, if class time allowed or as an assignment, for students to look at some scatter plots using either the graphing calculator or graph paper:

- n vs. P
- n vs. s
- n vs. area of each triangle
- n vs. area of the regular polygon

From the graphs the students could visually see that each of these quantities was approaching the values mentioned above from the class activity.

These two activities begin to help the students see how various quantities change in relationship to each other and that different quantities in the same activity can approach different values. Through an activity such as this students begin to gain an insight into the concept of a limit– a very foundational concept to Calculus, but one which can be view graphically and numerically through a geometric activity.

FINDING A FORMULA FOR THE AREA OF A CIRCLE

Use the attached regular polygons to complete the following information.
First draw one representative triangle in each polygon using two consecutive vertices and the center of the circumscribed circle. Do not draw any triangles in the circle.

n = number of sides
r = length of the radius of circumscribed circle
a = length of the apothem

s = length of the one side of the polygon
P = perimeter of the regular polygon
A_Δ = area of the representative triangle
A_{POLY} = area of the polygon

n	**r**	**a**	**s**	**P**	A_Δ	A_{POLY}

What observations can you make from the chart?

As n increases what happens with the length of the radius of the circumscribed circle?

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As n increases what happens with the length of the apothem?

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As n increases what happens with the length of side of the regular polygon?

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As n increases what happens with the length of the perimeter of the regular polygon?

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As n increases what happens to the area of the representative triangle?

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As n increases what happens to the area of the polygon?

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Think about what value each of the above statements approaches. Return to the above questions and write down the value they approach. What formula could be used to find the area of the regular polygon?________________________

Use this formula to generate a formula for the area of a circle, thinking about how the value of a changes and the value of P changes and n increases to a very large number.