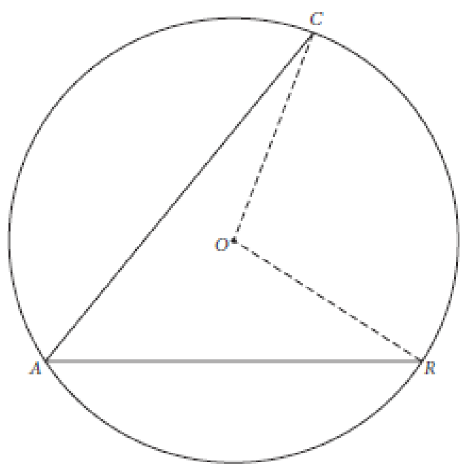
Engaging Students in Visualizing and Conjecturing in Geometry



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The van Hiele Levels of Geometric Understanding

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A husband-and-wife team of Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof, noticed the difficulties that their students had in learning geometry. These observations led them to develop a theory involving levels of thinking in geometry that students pass through as they progress from merely recognizing a figure to being able to write a formal geometric proof. Their theory explains why many students encounter difficulties in their geometry course, especially with formal proofs. The van Hieles believed that writing proofs requires thinking at a comparatively high level, and that many students need to have more experiences in thinking at lower levels before learning formal geometric concepts.

Frequently Asked Questions

Q. What are the van Hiele levels of geometry understanding?

A. There are five levels, which are sequential and hierarchical. They are:

Level 1 (Visualization): Students recognize figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.

Level 2 (Analysis): Students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.

Level 3 (Abstraction): Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction, however, is not understood.

Level 4 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems.

Clements and Battista (1992) also proposed the existence of Level 0, which they call *pre-recognition*. Students at this level notice only a subset of the visual characteristics of a shape, resulting in an inability to distinguish between figures. For example, they may distinguish between triangles and quadrilaterals, but may not be able to distinguish between a rhombus and a parallelogram.

Q. I've seen other numbering schemes for the levels. Why?

A. In their original works, the van Hieles numbered the levels from 0 to 4. Americans started numbering the levels from 1 to 5 instead. This scheme allows for the pre-recognition level to be called Level 0. Pierre van Hiele's recent works describe three levels of thought rather than five. This article uses the 1 to 5 numbering scheme.

Q. Is the development of geometric understanding related to age or maturation? experience? instruction?

A. Progress from one level to the next level is more dependent on educational experiences than on age or maturation. Some experiences can facilitate (or impede) progress within a level or to a higher level.

Q. What are the phases of learning?

A. According to the van Hieles, a student progresses through each level of thought as a result of instruction that is organized into five phases of learning. The phases are described below.

Information: Through discussion, the teacher identifies what students already know about a topic and the students become oriented to the new topic.

Guided orientation: Students explore the objects of instruction in carefully structured tasks such as folding, measuring, or constructing. The teacher ensures that students explore specific concepts.

Explicitation: Students describe what they have learned about the topic in their own words. The teacher introduces relevant mathematical terms.

Free Orientation: Students apply the relationships they are learning to solve problems and investigate more open-ended tasks.

Integration: Students summarize and integrate what they have learned, developing a new network of objects and relations.

A student may need to cycle through some of the five phases more than once with a particular topic.

Q. Can a student skip levels?

A. According to the van Hieles'model, a student cannot achieve one level of understanding without having mastered all the previous levels. Research in the United States and other countries supports this view with one exception. Some mathematically talented students appear to skip levels, perhaps because they develop logical reasoning skills in ways other than through geometry.

Q. What if the teacher is thinking at a different van Hiele level than the students?

A. This situation is common. Most high school geometry teachers think at the fourth or fifth van Hiele level. Research indicates that most students starting a high school geometry course think at the first or second level. The teacher needs to remember that although the teacher and the student may both use the same word, they may interpret it quite differently. For example, if a student is at the first level, the word "square" brings to mind a shape that looks like a square, but little else. At the second level, the student thinks in terms of the properties of a square, but may not know which ones are necessary or sufficient to determine a square. The student may feel that in order to prove that a figure is a square, all the properties must be proved. The teacher, who is thinking at a higher level, knows not only the properties of a square, but also which ones can be used to prove that a figure is a square. In fact, the teacher may think of several different ways to show that a figure is a square, since the teacher knows the relationships between the various properties and can determine which properties are implied by others. The teacher must evaluate how the student is interpreting a topic in order to communicate effectively.

Q. What happens if a teacher tries to teach at a level of thought that is above a student's level?

A. Generally, the student will not understand the content that is being taught. Usually, the student will try to memorize the material and may appear to have mastered it, but the student will not actually understand the material. Students may easily forget material that has been memorized, or be unable to apply it, especially in an unfamiliar situation.

Q. Will a student be at the same level of geometric understanding in all content strands?

A. Not necessarily. If a student has done more work with triangles than with quadrilaterals, he or she may think about triangles in a more sophisticated way than he or she would about an unfamiliar figure such as a trapezoid. Once a student has achieved a certain level of thought in one content strand, however, it is easier for him or her to think at that level in other areas, because he or she is accustomed to seeking out relationships between figures and between properties.

Q. What is the role of language in learning geometry?

A. Language plays an important role in learning. As indicated above, each level of thought has its own language and its own interpretation of the same term. Discussing and verbalizing concepts are important aspects of the Information, Explicitation, and Integration phases of learning. Students clarify and reorganize their ideas through talking about them.

Q. How can I assess a student's van Hiele level?

A. There are tests that can be used to assign a van Hiele level. Within a classroom, however, it is more practical for a teacher to assess a student's van Hiele level by analyzing his or her responses to specific geometric tasks. For example, a teacher can observe how a student uses geometric language and determine the student's level of thinking about triangles by analyzing his or her responses to the triangle sorting task in Section 2.2 of *Geometry: Explorations and Applications.*

Q. What are the implications of the van Hiele theory for my instructional practices?

A. The van Hiele theory indicates that effective learning takes place when students actively experience the objects of study in appropriate contexts, and when they engage in discussion and reflection. According to the theory, using lecture and memorization as the main methods of instruction will not lead to effective learning. Teachers should provide their students with appropriate experiences and the opportunities to discuss them.

Teachers can assess their students' levels of thought and provide instruction at those levels. The teacher should provide experiences organized according to the phases of learning to develop each successive level of understanding.

In *Geometry: Explorations and Applications,* activities at the intermediate levels are given to help students develop their understanding of figures and properties. Throughout the book, fundamental experiences and opportunities for discussion and reflection help develop successive levels of understanding. When proof is introduced, the students have had the important experiences necessary for it.

Q. What is the relationship of the NCTM *Curriculum and Evaluation Standards for School Mathematics* to the van Hiele theory?

A. Although it doesn't mention the van Hiele theory specifically by name, the *Curriculum and Evaluation Standards* (1989) states:

> Evidence suggests that the development of geometric ideas progresses through a hierarchy of levels. Students first learn to recognize whole shapes and then to analyze the relevant properties of shape. Later they can see relationships between shapes and make simple deductions. Curriculum development and instruction must consider this hierarchy. (p. 48)

The *Curriculum and Evaluation Standards* is consistent with the methodology advocated by the van Hiele model, especially the phases of learning.

> The Curriculum Standards present a dynamic view of the classroom environment. They demand a context in which students are actively engaged in developing mathematical knowledge by exploring, discussing, describing, and demonstrating. Integral to this social process is communication. Ideas are discussed, discoveries shared, conjectures, confirmed, and knowledge acquired through talking, writing, speaking, listening, and reading. (p. 214)

Q. What are some further references I can check for more information about the van Hiele levels?

A. Here is a list of some of the many materials that are available in professional journals and publications.

Burger, W. and J. Shaughnessy, "Characterizing the van Hiele Levels of Development in Geometry." *Journal for Research in Mathematics Education* 17: pp. 31–48.

Clements, D. and M. Battista, "Geometry and Spatial Reasoning." In D. Grouws, ed. *Handbook of Research on Mathematics Teaching and Learning*, New York: Macmillan Publishing Co., 1992.

Crowley, M. "The van Hiele Model of the Development of Geometric Thought." In M. Lindquist, ed., *Learning and Teaching Geometry, K–12*, 1987 Yearbook. Reston: National Council of Teachers of Mathematics, 1987.

Fuys, D., D. Geddes, and R. Tischler, "The van Hiele Model of Thinking in Geometry Among Adolescents." *Journal for Research in Mathematics Education Monograph*, 3. Reston: National Council of Teachers of Mathematics, 1988.

Mason, M. (in press). "The van Hiele Model of Geometric Understanding and Mathematically Talented Students." *Journal for the Education of the Gifted*.

Mason, M. and S. Moore (in press). "Assessing Readiness for Geometry in Mathematically Talented Middle School Students." *Journal of Secondary Gifted Education*.

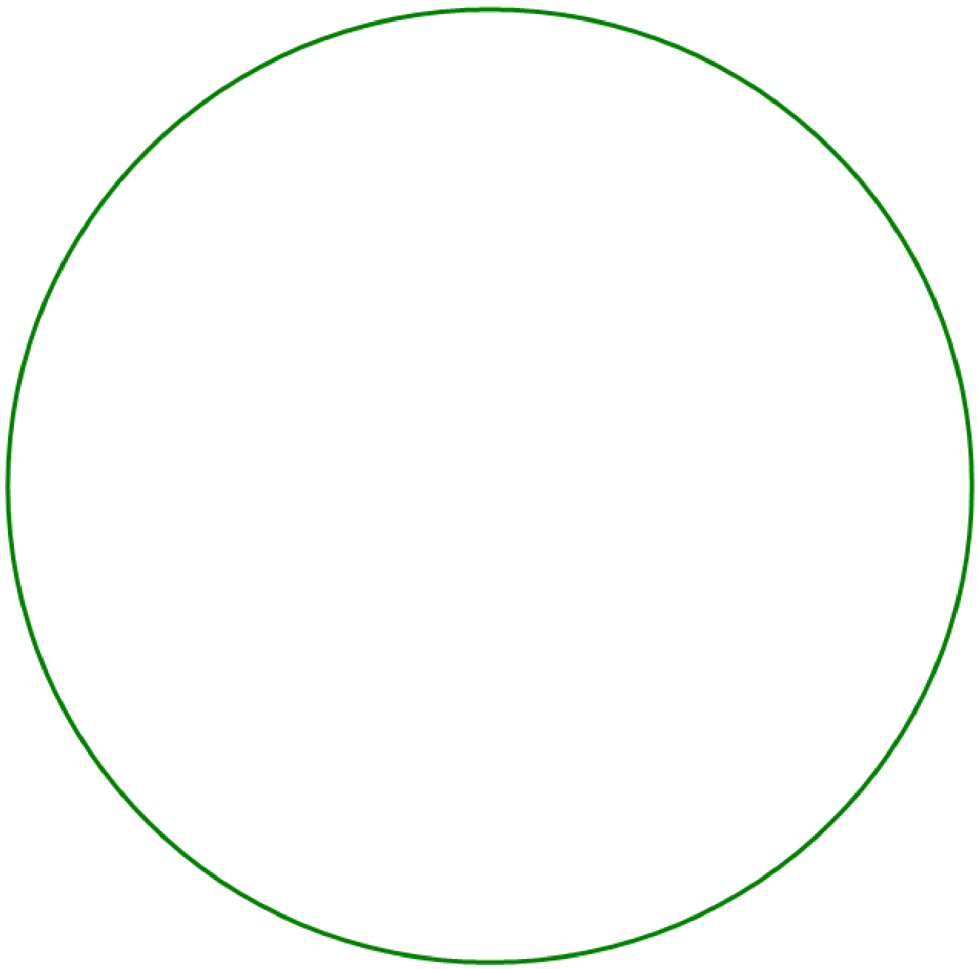
Mayberry, J. "The van Hiele Levels of Thought in Undergraduate Preservice Teachers." *Journal for Research in Mathematics Education*, 14: pp. 58–69

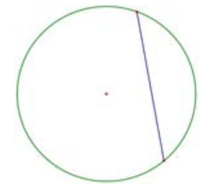
Senk, S. "Van Hiele Levels and Achievement in Writing Geometry Proofs." *Journal for Research in Mathematics Education*, 20: pp. 309–321.

Teppo, A. "Van Hiele Levels of Geometric Thought Revisited." *Mathematics Teacher*, 84: pp. 210–221.

Usiskin, Z. *Van Hiele Levels and Achievement in Secondary School Geometry*. (Final Report of the Cognitive Development and Achievement in Secondary School Geometry Project.) Chicago, Illinois: University of Chicago, 1982.

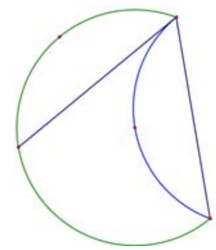
Folding A Circle





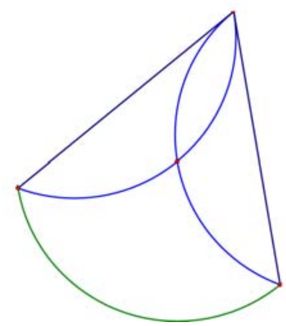
Have students cut out the circle. Ask students to fold the edge of the circle (circumference) until it hits the center.

Open the circle and describe the name of the line segment. (Chord)



Fold the edge of the circle again so it hits the center of the circle.   Be sure the one fold touches the previous fold.

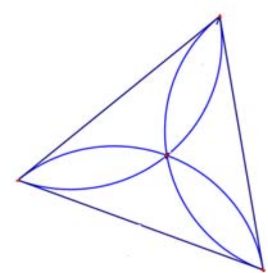
Open the folds. Can you name the figures you see in the circle. (Chord and inscribed angle).



Fold the edge of the circle one final time.

Open the circle and describe what you see. (Three congruent chords, three inscribed angles, three congruent angles, equilateral triangle.)

How big are the three angles? How big are the three arcs?



Do you see any radii? (No) Explain why not. No line segments begin at the center of the circle.

Make a conjecture on how you can find the size of an inscribed angle. (One half of the marked arc.)

Fold each vertex of the triangle to the center of the circle. Create the three dimensional figure using this material. What is the figure called? (Tetrahedron - four faced pyramid)

If the triangle has an area of 4 square units, what is the area of each triangle? (1 square unit)

How do the lengths of the sides of these small triangles compare to the length of the sides of the larger triangle? (½) How do the area of the two triangles compare? (1/4)

How do the perimeters of the two triangles compare? (1:2) Give a convincing argument for this ratio?

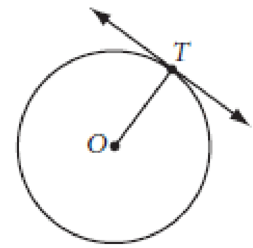
A point has been identified on each side of the original equilateral triangle. What is the name of this point? (Midpoint)

Line segments connect these midpoints. How do these segments compare to the sides of the large equilateral triangle. (Parallel and ½ the length). Give support for your answer.

The large equilateral triangle has been subdivided into what? (Four congruent equilateral triangles) Fold each vertex of the original equilateral triangle to the midpoint of the corresponding side of the middle equilateral triangle. Form the three dimensional figure with this net. What is it called? (A truncated tetrahedron)

Gather 20 of these tetrahedrons to form an icosahedron.

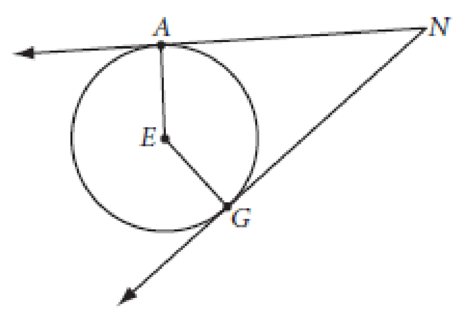
What is special about a tangent line to a circle?



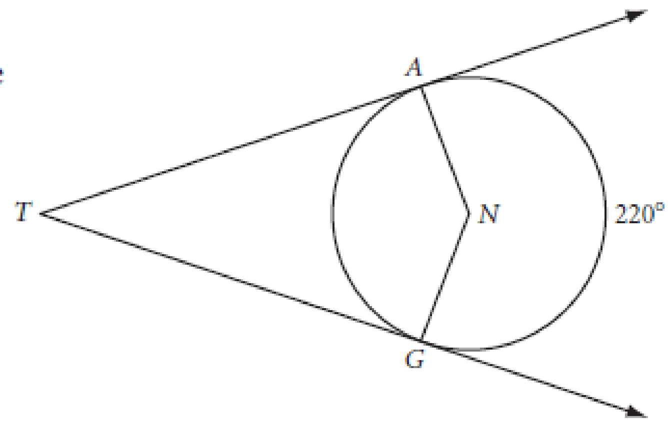
Step 1	Construct a large circle. Label the center *O*.
Step 2	Using your straightedge, draw a line that appears to touch the circle at only one point. Label the point *T*. Construct $\overline{OT}$.
Step 3	Use your protractor to measure the angles at *T*. What can you conclude about the radius $\overline{OT}$ and the tangent line at *T*?
Step 4	Share your results with your group.

What is special about two tangent segments from one point outside a circle?

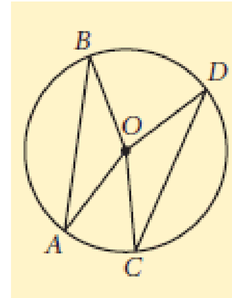
Step 1	Construct a circle. Label the center *E*.
Step 2	Choose a point outside the circle and label it *N*.
Step 3	Draw two lines through point *N* tangent to the circle. Mark the points where these lines appear to touch the circle and label them *A* and *G*.
Step 4	Use your compass to compare segments *NA* and *NG*. Segments such as these are called **tangent segments.**
Step 5	Share your results with your group.



In the figure at right, $\overrightarrow{TA}$ and $\overrightarrow{TG}$ are both tangent to circle N. If the major arc formed by the two tangents measures 220°, find the measure of $\angle T$.



What happens when we draw two congruent chords?

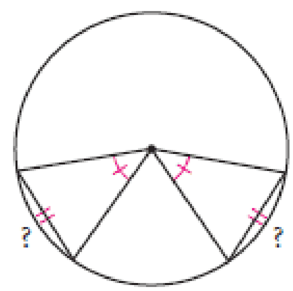


Step 1 Construct a large circle. Label the center O. Construct two congruent chords in your circle. Label the chords $\overline{AB}$ and $\overline{CD}$, then construct radii $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$.

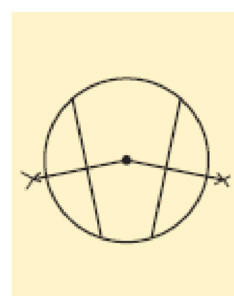
Step 2 With your protractor, measure $\angle BOA$ and $\angle COD$. How do they compare? Share your results with others in your group.

Step 3 How can you fold your circle construction to check the conjecture?

Step 4 Recall that the measure of an arc is defined as the measure of its central angle. If two central angles are congruent, their intercepted arcs must be congruent.



How is a perpendicular through the center of a circle related to any chord?

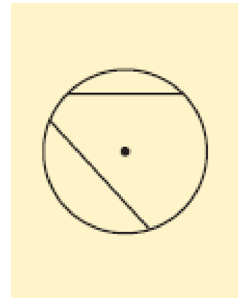


Step 1 Construct a large circle and mark the center. Construct two nonparallel congruent chords. Then construct the perpendiculars from the center to each chord.

Step 2 How does the perpendicular from the center of a circle to a chord divide the chord? Copy and complete the conjecture.

Step 3 Compare the distances (measured along the perpendicular) from the center to the chords. Are the results the same if you change the size of the circle and the length of the chords? State your observations as your next conjecture.

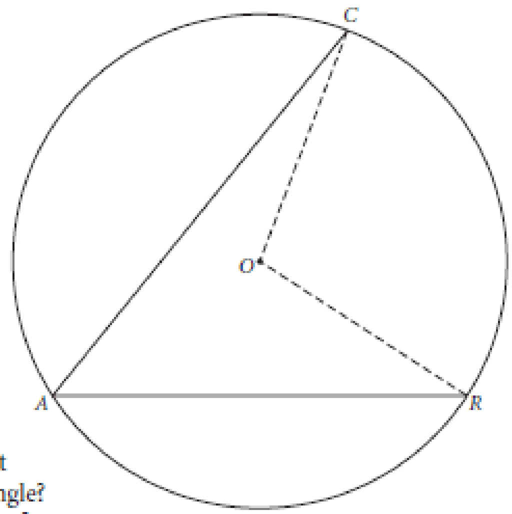
How can I find the center of a circle?



Step 1 | Construct a large circle and mark the center. Construct two nonparallel chords that are not diameters. Then construct the perpendicular bisector of each chord and extend the bisectors until they intersect.

Step 2 | What do you notice about the point of intersection? Compare your results with the results of others near you.

Draw circle O. Draw a central angle COR. Draw a corresponding inscribed angle CAR.

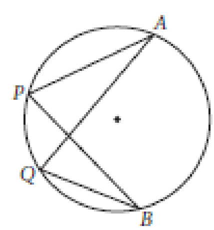


Step 1 Measure $\angle COR$ with your protractor to find $m\overset{\frown}{CR}$, the intercepted arc. Measure $\angle CAR$. How does $m\angle CAR$ compare with $m\overset{\frown}{CR}$?

Step 2 Construct a circle of your own with an inscribed angle. Draw and measure the central angle that intercepts the same arc. What is the measure of the inscribed angle? How do the two measures compare?

Step 3 Share your results with others near you.

What will happen if we have two inscribed angle that open up onto the same arc?

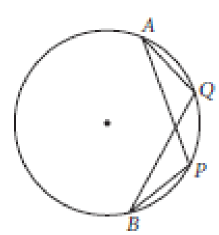


Step 1 Construct a large circle. Select two points on the circle. Label them A and B. Select a point P on the major arc and construct inscribed angle APB. With your protractor, measure $\angle APB$.

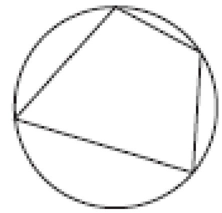
Step 2 Select another point Q on $\overset{\frown}{APB}$ and construct inscribed angle AQB. Measure $\angle AQB$.

Step 3 How does $m\angle AQB$ compare with $m\angle APB$?

Step 4 Repeat Steps 1–3 with points P and Q selected on minor arc AB. Compare results with your group.



What is special about a quadrilateral that is inscribed in a circle?



Step 1 Construct a large circle. Construct a cyclic quadrilateral by connecting four points anywhere on the circle.

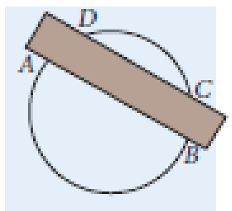
Step 2 Measure each of the four inscribed angles. Write the measure in each angle. Look carefully at the sums of various angles. Share your observations with students near you.

What happens if we construct two parallel secant lines?

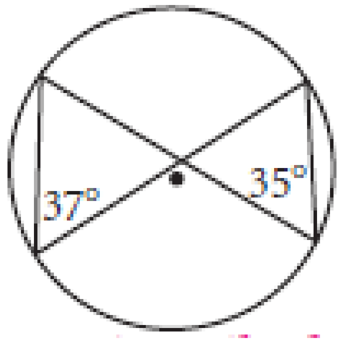
Step 1 On a piece of patty paper, construct a large circle. Lay your straightedge across the circle so that its parallel edges pass through the circle. Draw secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ along both edges of the straightedge.

Step 2 Fold your patty paper to compare $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$. What can you say about $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$?

Step 3 Repeat Steps 1 and 2, using either lined paper or another object with parallel edges to construct different parallel secants. Share your results with other students.



Explain what is wrong with this picture.



Explain why $\overset{\frown}{AC} \cong \overset{\frown}{CE}$.