Looking at New Approaches to a Few Pre-Calculus and Calculus Topics

Unit Circles
Introducing Series and Convergence

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Visualizing a Unit Circle

1. Cut out a large circle or use a large paper plate.
2. Fold the circle in half to get one diameter. (Fold 1)
3. Fold the circle in half again to get another diameter (Fold 2).
4. Draw in the x- and y-axis. Label the radius as 1 unit. Label the coordinates of the points and their radians. Complete the chart below.

Coordinates	(,)	(,)	(,)	(,)
Radians				

5. Refold the circle to position of Fold 2. Fold the quarter circle again in half. Draw in this lines. Each of these new lines is at $\frac{\pi}{4}$ from the original x- or y-axis. In the first quadrant, drop a vertical line segment from the endpoint of the radius. What special triangle has been formed? Use this to calculate the sides of the right triangle. This should help you name the coordinates for this endpoint. Thinking about reflections over the x- and y-axis, name the coordinates of the endpoints for the other radii.

6. Label the coordinates of the points. Label their radians also. Complete the chart below.

Coordinates	(,)	(,)	(,)	(,)
Radians				

7. Refold the circle to the position of Fold 2. Once you have the quarter of a circle, fold the quarter into thirds. This can be done by folding one piece inside the other until you have 3 equal pieces overlapping. Open the circle or plate. Notice how each quadrant has been divided. Notice there are not 4 equal pieces in each quadrant. Create right triangles in the first quadrant from the endpoint of each radius. What special right triangle has been formed? Calculate the length of each side of the special triangle.

8. Draw in these 4 lines. Label these new points. Identify the radian measure also. Complete the chart below for these eight points.

Coordinates	(,)	(,)	(,)	(,)
Radians				
Coordinates	(,)	(,)	(,)	(,)
Radians				

9. Recall that the coordinate of each endpoint on the unit circle is (cos A, sin A). Use this unit circle to help you recall the values in the table for Trigonometric Values.

Radian Measure	Sine	Cosine

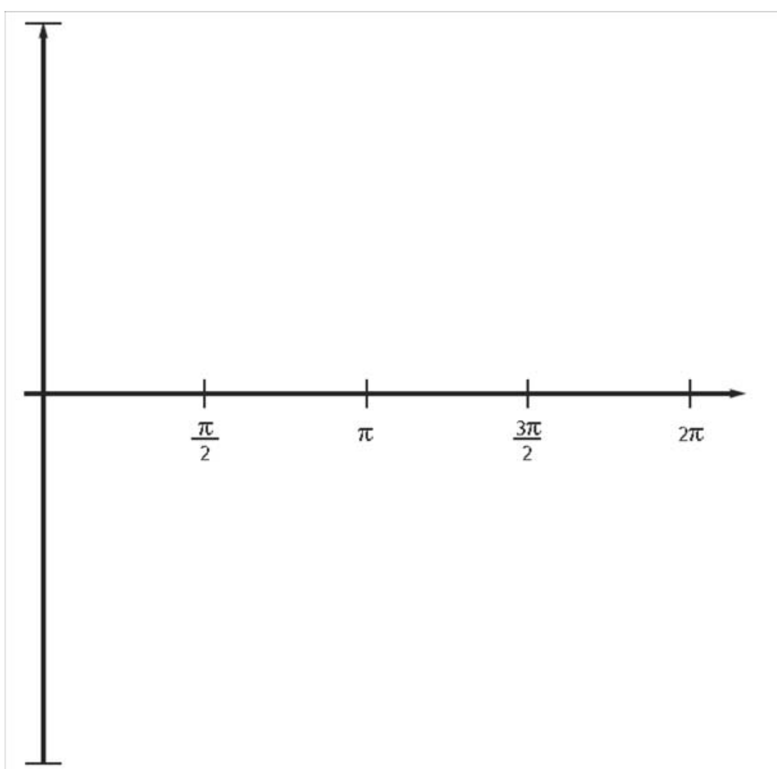
Creating Graphs for Sin and Cos Functions

Place the Unit Circle Graph Generator Template in your communicator.
On the unit circle, label the y coordinate at each of the indicated points.

Graphing the sine function

As you move around the circle note the radian measure for each point. You will associate each radian measure with its y-coordinate. Move to the graph at the right and at each radian value, mark the y-coordinate as the height of the function. For example at $\frac{\pi}{6}$ the y-coordinate would be $\frac{1}{2}$. Therefore at the right graph the point $\left(\frac{\pi}{6},\frac{1}{2}\right)$. You can visually pick up these values from the unit circle. Complete a graph for all 16 points along the unit circle. When you are done you have graphed the sine graph.

Draw a sketch of this graph below:



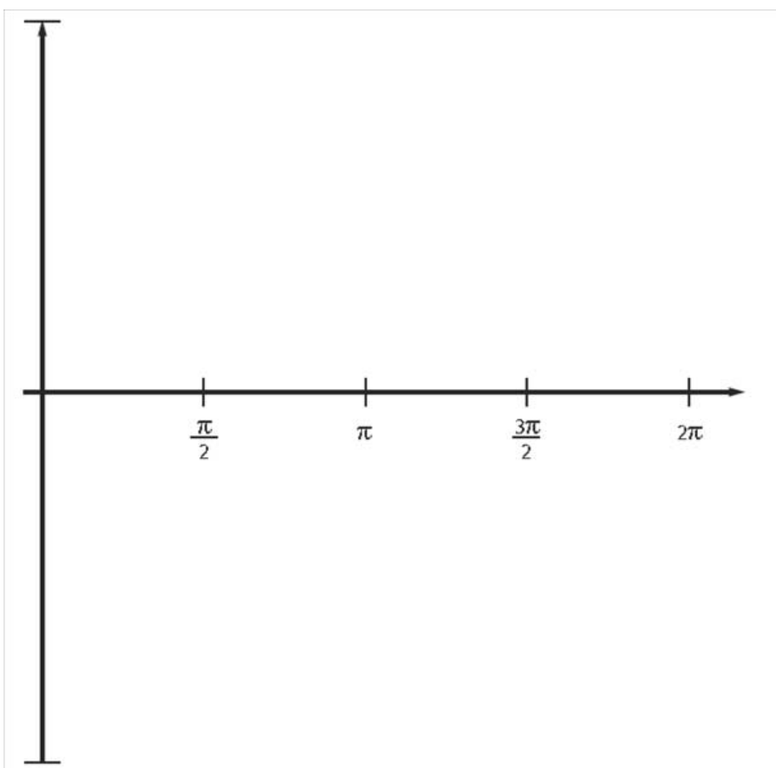
Graphing the cosine function

This graph is little harder to graph since the x-coordinate from the unit circle must be graphed as the height of the new point on the graph.

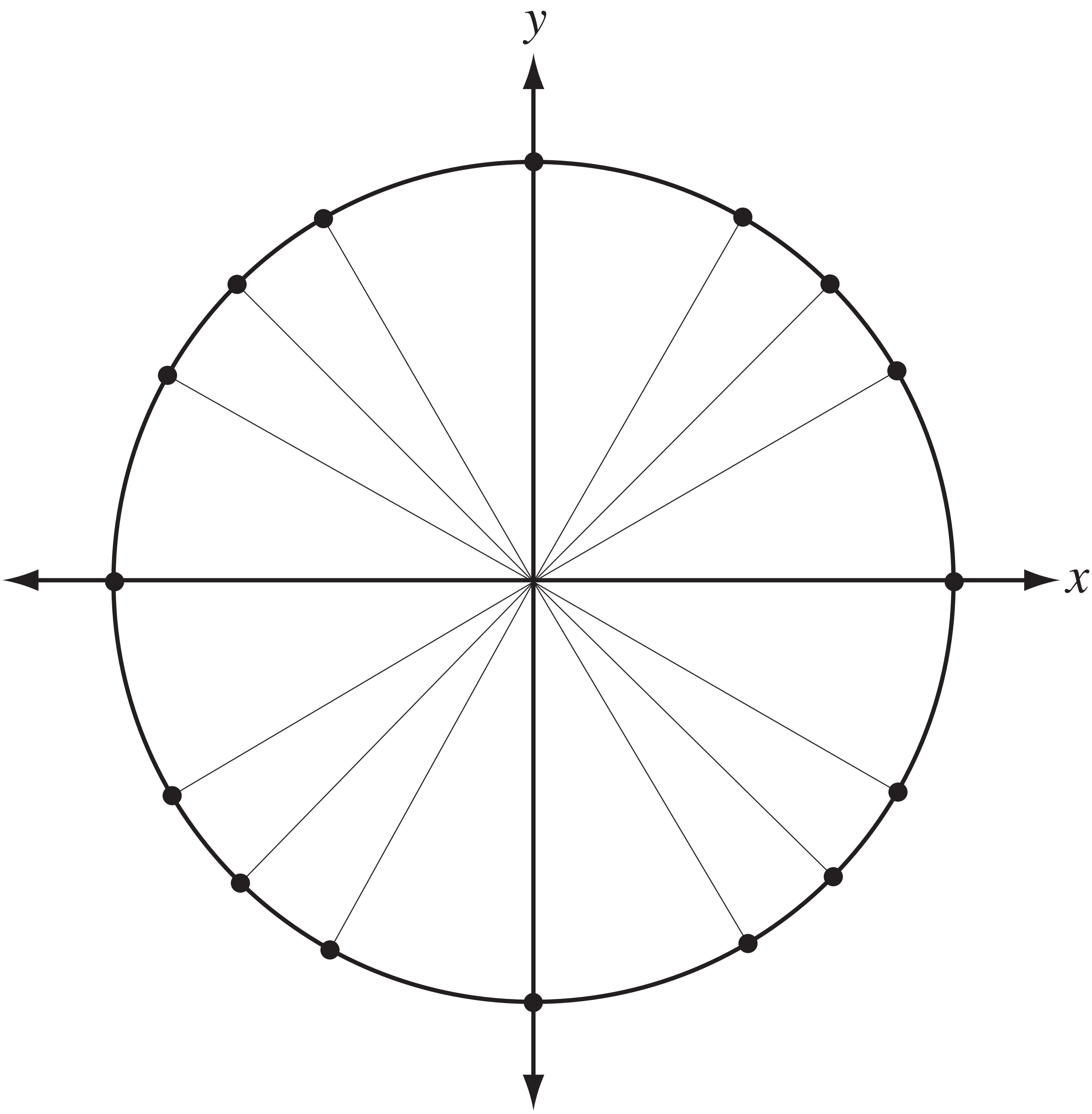
As you move around the circle note the radian measure for each point. You will associate each radian measure with its x-coordinate. Move to the graph at the right and at each radian value, mark the x-coordinate as the height of the function. (But remember that the x- and y-coordinates switch values.)

This means that if you need to graph a height of $\frac{\sqrt{3}}{2}$, all you need to do is find a point of the unit circle that has a y-value of $\frac{\sqrt{3}}{2}$. At $\frac{\pi}{6}$ the x-coordinate would be $\frac{\sqrt{3}}{2}$. Therefore at the right graph the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.

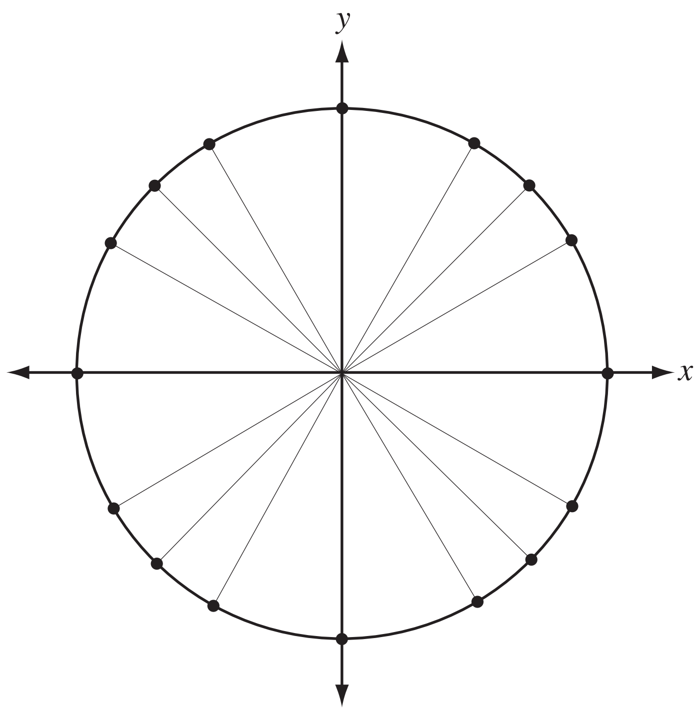
To get the height of $\frac{\sqrt{3}}{2}$ you can use the height at the $\frac{\pi}{3}$ point on the unit circle. You can visually pick up these values from the unit circle. Complete a graph for all 16 points along the unit circle. When you are done you have graphed the cosine graph.

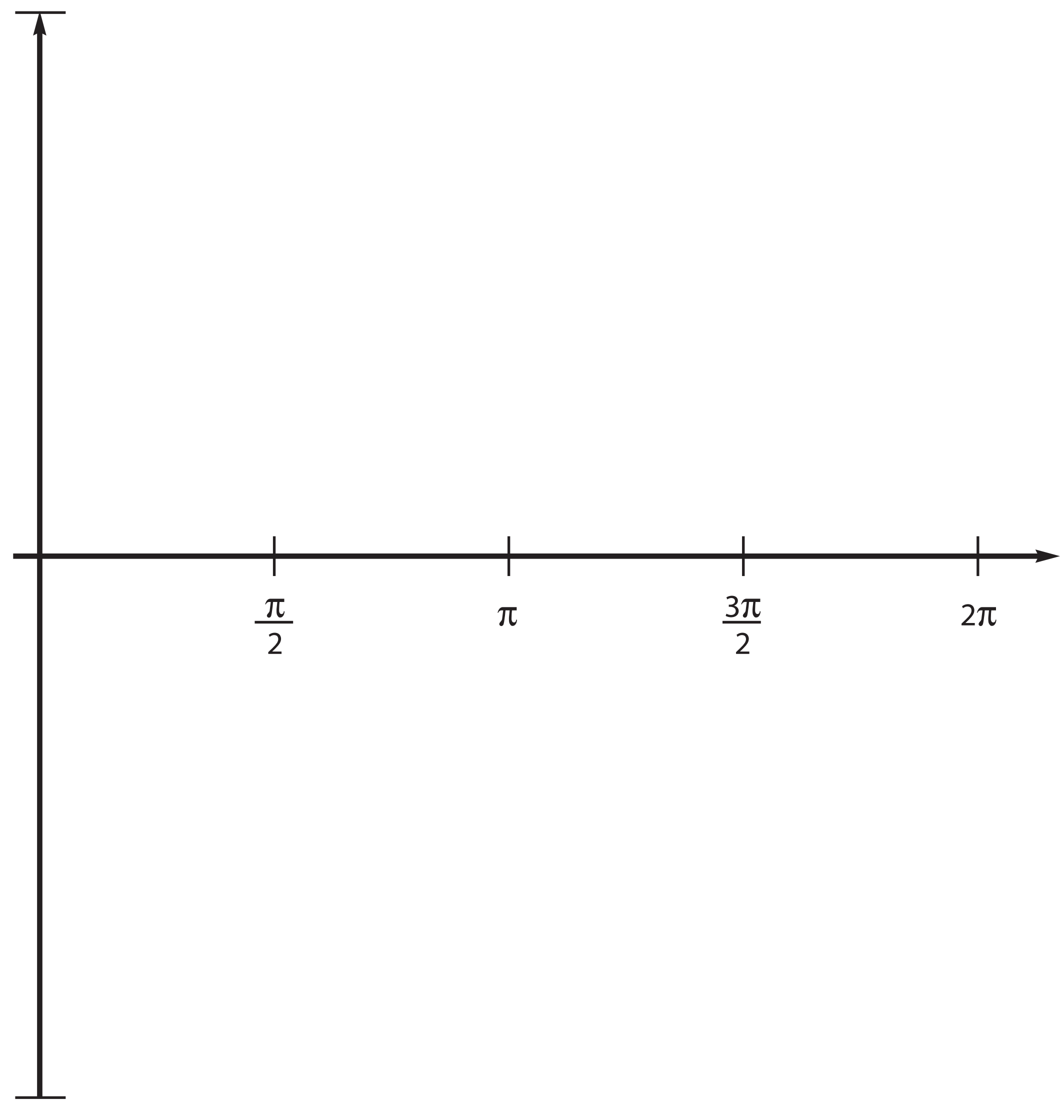


Un-Labeled Unit Circle



Unit Circle Graph Generator (Radians)





Creating Graphs of Common Trig Functions with the Unit Circle on the TI-84

Turn on the calculator. Change the mode to Parametric equations.
Enter the equations $x_1t = \cos t, y_1t = \sin t$ into the first set of equations.
Set the window to have t start at 0 and go to 2π at steps of 0.1. Have $-1.5 \le x \le 2\pi$ and $-2.566 \le y \le 2.566$. Graph this pair of functions and you should notice a unit circle. Trace around the circle to various locations and read the coordinates. If you want a specific point, type that number in after you press trace.

Let's look at a graph y=sin x.

Enter a second set of parametric equations: $x_2t = t, y_2t = \sin t$. You should see the unit circle and a new graph of y = sin x. Press TRACE. To reach a specific t value, type in the value after you have pressed TRACE. As you locate a point on the unit circle, use the up or down arrows to move to the graph. You will find the corresponding points between the unit and circle and the sine graph.

Record the values of the sin x for the following x values:

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
sin x							

X	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
sin x							

X	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π				
sin x							

As you move around the circle, notice where some of these points appear on the sine graph.

Enter a third set of parametric equations: $x_3t = t, y_3t = \cos t$. You should see the unit circle and a new graph of y = sin x. Press TRACE. To reach a specific t value, type in the value after you have pressed TRACE. As you locate a point on the unit circle, use the up or down arrows to move to the graph. You will find the corresponding points between the unit and circle and the sine graph.

Record the values of the sin x for the following x values:

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
cos x							

X	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
cos x							

X	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π				
cos x							

As you move around the circle, notice where some of these points appear on the sine graph.

You can continue to study other trigonometric functions by replacing the second of the two equations with tan t, cot t, sec t, and csc t.

Introducing Series and Convergence

Polynomials can be built that resemble a function in a small interval.

Recall that if a line T(x) is drawn tangent to a function f(x) at a point x = a two properties are true

a. T(a) = f(a) (there is a point in common)
b. T′(a) = f′(a) (the slope is the same)

These ideas can be translated to property of local linearity: A differentiable function, over a small interval, resembles a straight line. This means that we can approximate values on a function at a point near x = a, but finding values along a tangent line.

Suppose $f(x) = x^2$. Write a equation for the tangent line at x = 2.

T(x) =

Use your graphing calculator to graph a picture of f(x) and T(x) in a small window near x=2.

Create a set of table values near x = 2 with ΔtI = 0.1

What do you notice from the graph and the table?

The straight lines does not perfectly match up with the parabola, but gives us values very close to those on the function when we are near the point of tangency. Suppose we think of a different function such as y=ln x.

Again find the tangent line to y=ln x at x = 2 and graph the tangent line along with it's tangent line at x = 2.

T(x) =

Would a parabola fit the curve better than the straight line at x = 2? Recall that the tangent line was of the form $T(x) = f'(2)(x-2) + f(2)$ or, in general, $T(x) = f'(a)(x-a) + f(a)$ A general parabola that could be: $y = c_2(x-a)^2 + c_1(x-a) + f(a)$. Notice that at x=a, the value on the parabola would equal f(a).

Let's solve for c_1 and c_2. This parabola and function f(x) should have the same tangent line at x=a so the derivative of f and the first derivative of the parabola should be equal.

so at x = a

c_1 =

Since the parabola and the functions should curve the same way their concavity should be the same so

c_2 =

Therefore, the parabola that fits the function f(x) would have the form

P(x) =____

Let's find the parabola that fits the function y=ln x.

P(x) = ______________________________________

Graph the function, it's tangent line and it's parabola at x=2. Look at a set of table values for the three functions near x = 2.

What do the graph and the table values illustrate?

Continuing this process, the cubic polynomial that fits the function f(x) would be

$$C(x) = c_3(x-a)^3 + c_2(x-a)^2 + c_1(x-a) + f(a)$$

Solve for the values of c_1, c_2, and c_3, using similar reason that we used with the parabola.

Study patterns that you see developing to write the polynomial of degree four that fits the function.

P(x) =

These equations are called Taylor polynomials or various degrees. Enter each new polynomial in the calculator and view the graphs and tables associated with each. Each time you enter a new polynomial notice you can start with the previous equation and add only one new term. This will make it easier to enter the new equation.

What do you notice about the Taylor polynomials?

1. As the degree of the polynomial increases the graph of the new polynomial comes closer to the graph of the function than the previous polynomial.

2. The polynomials exist outside the domain of the function f.

3. The interval where the graphs are close to the function f is limited.

Write out a general Taylor polynomial for a function f.

What would happen if we continued to add more terms to the Taylor polynomial to create a series?

What would happen if we formed a series with an infinite number of terms? We call this a Taylor series.

Exercises: For each function, graph the function near x = a and write several Taylor Polynomials. View the graphs and table values for each.

1. $y = e^x$ near $x = 0$

2. $y = \cos x$ near $x = 0$

Let's study $y = \sin x$ near $x = 0$

Write out the Taylor polynomials of degree 1, 3, 5, 7, and 9.

$T_1(x) =$

$T_3(x) =$

$T_5(x) =$

$T_7(x) =$

$T_9(x) =$

Graph each Taylor polynomial against the original function.

Did you notice that as you added more terms on the Taylor polynomial for y = sin x the interval where the curves matched up seemed to get wider.

Write out the Taylor series for y=sin x.

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Notice as you add terms to the Taylor polynomial the Taylor polynomial approaches the function y = sin x.

Set up a new equation that compares the Taylor polynomial with the original function: $y = |y1 - T(x)|$. Set up a window : $-4 \le x \le 4$ and $-0.2 \le y \le 0.2$. If $T(x)=T_3(x)$ then this equation is describing the error between the function y = sin x and the $T_3(x) = -\frac{1}{3!}(x-0)^3 + 1(x-0) + 0$. Keep changing the Taylor polynomial and notice what happens to this graph.

The interval where the Taylor series comes close to the function is called an interval of convergence. We say that the series converges when the x values are within the interval. When the x values are outside the interval the series diverges. Interval of convergence are centered around the point x = a and may be opened, closed, or half open.

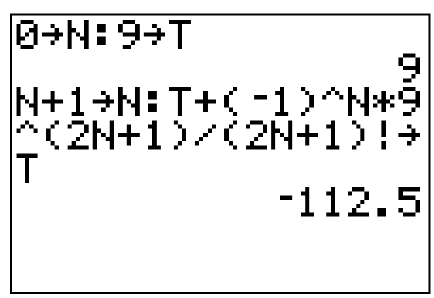
Describe what this graph is telling you about the two functions.

Remember the Taylor polynomial for y= sin x is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. How many terms of the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ are required to approximate sin 9 accurately to the third decimal place?

First find the sin 9 on your calculator:

sin 9 = .4121184852

Now let's set up a recursive set of steps that calculated one additional term every time we press enter.



Line 1: 0 sto N: 9 sto T Press ENTER
Line 2: N+1 sto N: T+(-1)^N*9^(2N+1)/(2N+1)! Press ENTER

The first time you press enter you have the value of third degree Taylor polynomial at x = 9. This is a value along a cubic equation that closely fits y=sin x near x=0.

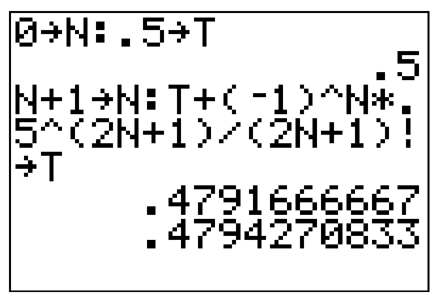
Each time you press ENTER you are adding one more term to the polynomial. Keep track of the degree of the polynomial, the value, and stop when you have accuracy to three decimal places.

Degree of Taylor Polynomial	Approximate value at x = 9 along the polynomial

What happens when we pick a value of x nearer to x=0 such as x = 0.5

Rewrite the two lines to fit x = 0.5. Find the value for sin 0.5

sin .5 = .479426



Line 1: 0 sto N: .5 sto T Press ENTER
Line 2: N+1 sto N: T+(-1)^N*.5^(2N+1)/(2N+1)! Press ENTER

The first time you press enter you have the value of third degree Taylor polynomial at x = 0.5. This is a value along a cubic equation that closely fits y=sin x near x=0. What do you notice.