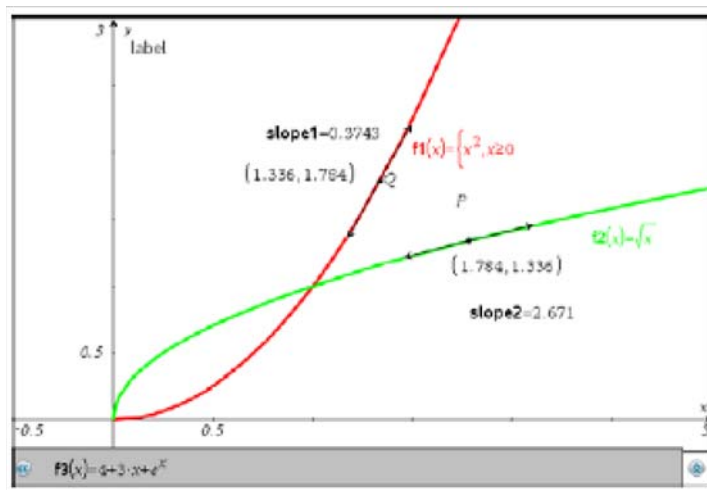


Studying Rates of Change with the Graphing Calculator



Red Bank Regional High School

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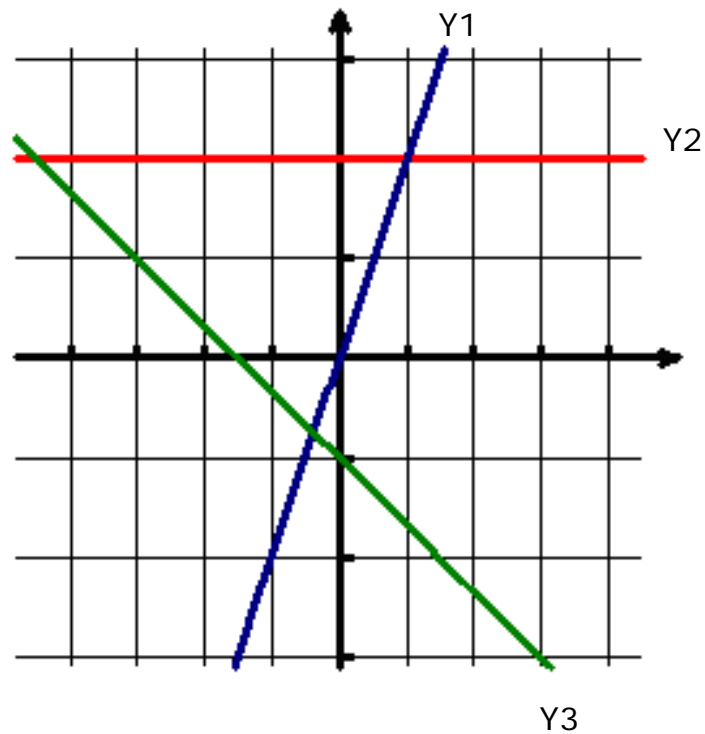
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Introduction to Local Linearity

There are three functions graphed on this set of axes.



1. Write a rule for each of the three lines. Give justification for why you wrote each equation.

Y1 = _____

Y2 = _____

Y3 = _____

2. Match each of the following with one of the graphs and justify your reasoning.

$$y = x^3 + 0.002 : \underline{\hspace{2cm}}$$

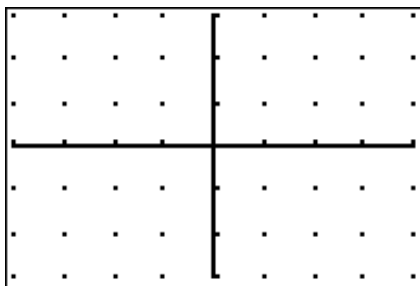
$$y = -\frac{2}{3}x - 0.001 : \underline{\hspace{2cm}}$$

$$y = \sin(2x) : \underline{\hspace{2cm}}$$

3. Enter each of these functions in your graphing calculator in a zoom 4 Decimal window. Record your sketch below.



4. Zoom in on the origin by resetting the window to $[-0.004, 0.004, 0.001, -0.003, 0.003, 0.001]$. Make a sketch of your graph.



5. What has happened to each of the graphs when you look at a very small window around the origin?

We say that a function is locally linear when we can make a curved line appear linear. Each straight line equation that you wrote for question 1 is called a linear approximation for these graphs at the point $x = 0$.

6. Graph the equation $y = \tan\left(\frac{x}{2}\right)$ on a zoom 4 decimal window. Zoom in to a very small window as you did in question 4. Write the equation of the line that can be used as the linear approximation for this function at $x = 0$.

Creating Linear Models for Functions at a point $(a, f(a))$

Goal: To create linear functions to model the behavior of nonlinear functions in a neighborhood of $x = a$ with the condition that we want our linear model to contain $(a, f(a))$ and have a slope that would make it appear to behave as $y = f(x)$ does at $x = a$.

Part I: Using tables to study the behavior of functions

All functions do not have constant rates of change ($\frac{\Delta y}{\Delta x}$ is not constant). But many functions have nearly constant rates of change when you study a function very close to particular x values. Create a table of values for each of the given functions near $x = a$. Refine your table until it reflects a constant rate of change. Use the information in the chart to write the linear function in point slope form that passes through $(a, f(a))$ with the approximate slope.

Example: $y = x^2$ at $x = 2$.

X	Y ₁	
1.997	3.988	
1.998	3.992	
1.999	3.996	
2	4	
2.001	4.004	
2.002	4.008	
2.003	4.012	
$X=2$		

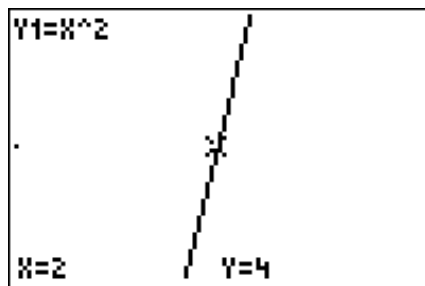
The point on the function is at $(2, 4)$. Studying the change in y over the change in x you can make the conclusion that the rate of change appears to be $\frac{\Delta y}{\Delta x} = \frac{.004}{.001} = 4$, so the equation of the linear function that models the behavior of $y = x^2$ at $x = 2$ is $y = 4(x - 2) + 4$

Write linear models for each of the functions at the given point $x = a$ using the tabular method.

- | | |
|---|--|
| <p>1. $y = x^3 + 5$ at $x = 3$</p> <p>3. $y = \sqrt{x} + 2$ at $x = 9$</p> <p>5. $y = \sqrt[3]{x-2} + 4$ at $x = 3$</p> | <p>2. $y = \sin 4x$ at $x = \frac{2\pi}{3}$</p> <p>3. $y = \ln x - 2$ at $x = 12$</p> <p>6. $y = e^{x^2}$ at $x = 1$</p> |
|---|--|

Part II: Using the graphical model to study the behavior of functions

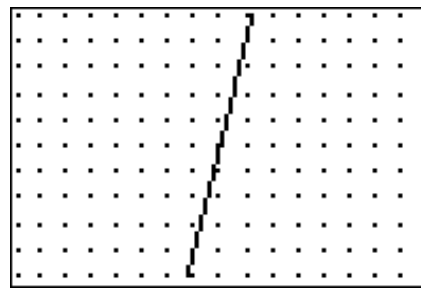
Example: $y = x^2$ at $x = 2$.



Zooming in on the point (2,4) your window twice should look like the one at the left. You can confirm that (2,4) is on the line and in your window by checking the value of the function at $x = 2$. By tracing you should notice that points on either side of (2,4) are about (1.996, 3.984) and (2.004, 4.017).

Calculating $\frac{\Delta y}{\Delta x} = \frac{.033}{.008} \approx 4$ confirms what we saw on the table values in part I so the equation of the linear function that models the behavior of $y = x^2$ at $x = 2$ is $y = 4(x - 2) + 4$

With a grid every 0.05 units the line looks like the straight line portion of the function has a slope of 4 (change of 4 in the y direction for a change of 1 in the x direction).



Write linear models for each of the functions at the given point $x = a$ using the graphical model.

- | | |
|---------------------------------------|---|
| 1. $y = x^3 - 5$ at $x = 2$ | 2. $y = \sin x$ at $x = \frac{2\pi}{3}$ |
| 3. $y = \sqrt{x} + 3$ at $x = 4$ | 3. $y = \ln x - 1$ at $x = 12$ |
| 5. $y = \sqrt[3]{x-1} + 3$ at $x = 2$ | 6. $y = 2^{x^2}$ at $x = 1$ |

An Exploration to Discover the Value for Derivative at a Point

Goal: To use tables to approximate the value of a $f'(a)$ on the function $f(x)$.

We know that $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$. Let's use the calculator to help us determine the value of $f'(a)$ given $f(x)$.

Let $f(x) = \sin x$ and let $x = 0$.

Enter $y_1 = \sin x$ and $y_2 = \frac{y_1(x) - y_1(0)}{x - 0}$ or $\frac{\sin x}{x}$

Create a table to study the values near $x = 0$.

X	Y2
-.003	1
-.002	1
-.001	1
0	ERROR
.001	1
.002	1
.003	1

Y2 = .999999833333

X	Y2
-4E-4	1
-3E-4	1
-2E-4	1
-1E-4	1
0	ERROR
1E-4	1
2E-4	1

Y2 = .999999998333

We know that the value at $x = 0$ is undefined. But looking at the two tables we see values get very close to 1 as we get very close to $x = 0$. Therefore I will make the conjecture that y_2 approaches 1 as x approaches a .

Use the table feature of the calculator to approximate the value of $f'(a)$ on the function $f(x)$.

- $y = e^x$ at $x = 1$
- $y = \frac{1}{x}$ at $x = 1$
- $y = \ln x$ at $x = 1$
- $y = |3x|$ at $x = 3$
- $y = \frac{1}{3}x^3 + 2$ at $x = 1$
- $y = \sqrt{x}$ at $x = 1$
- $y = \tan x$ at $x = \frac{\pi}{4}$
- $y = \frac{x^2 - 4}{x - 2}$ at $x = 3$.
- $y = 3^x$ at $x = 1$
- $y = \sqrt[3]{x}$ at $x = 8$

Discovering the Relationship between the Slopes at Corresponding Points on Inverse Functions

Using your TI-84 graphing calculator, enter $y_1 = (x^2)(x \geq 0)$ and $y_2 = \sqrt{x}$. Set the window to $[-.5,3,.5,-.5,3,.5,1]$. Make a graph of these two equations and then make a sketch of your picture at the right.



Step 1:

Use your graphing calculator to find the coordinate and the slope of y_1 at the point $x=0.4$. Record this point under y_1 on the *Charting Information About Slopes on Inverse Functions* chart under the column Point on f . Name the corresponding point that would be on y_2 . Use your calculator to find the slope of the functions f and f^{-1} at the given points. You can do this by using 2nd Draw: 5: tangent, make sure you are on y_1 and type in 0.4). Add this tangent line and the point to the sketch above. Repeat the drawing of a second tangent line on at the corresponding point (0.16,0.4) on y_2 at the point $x=0.16$. Remember to move to y_2 before selecting the point 0.16. Add this point and tangent line to y_2 in your sketch.

Record their slope in the appropriate columns on the *Charting Information About Slopes on Inverse Functions*.

Find 5 additional random points on f and their corresponding points on f^{-1} . Repeat the last procedure with these 5 points. Record the points and slope on your chart. You do not have to draw the tangent lines on the sketch.

After you have completed the first four columns for f and f^{-1} , find the product of the two slopes. What do you observe?

Step 2:

Let's see if this is true for other points along the graph of f and f^{-1} . Set up list 1 for 30 values between $1 \leq x \leq 3$ by moving to the top of L1 and entering the sequence: $\text{seq}(x,x,.1,3,.1)$. Ask the calculator to find the y values on the function f in L2 by moving to the top of L2 and typing $y1(L1)$. This should give you the 30 corresponding y values on the function f at the 30 values in L1. L1 should represent 30 x -values and L2 represents the 30 corresponding y -values on the function f .

Let's ask the calculator to find the slope of the function f at the values in L1 and the slope of the function f^{-1} at the value in L2. This corresponds with the same thing you did in step 1.

In y_3 type $\text{nderiv}(y1,x,x)$. If you are using the new operating system (2.53MP or above) for the ti-84 you can use alpha window and select nderiv and enter $y3 = \frac{d}{dx}(y1) \Big|_{x=x}$. In $y_4 = \text{nderiv}(y2,x,x)$ or $\frac{d}{dx}(y2) \Big|_{x=x}$. Turn these equation off by unhighlighting the equal signs. Then return to the lists and enter $L3=y3(L1)$ and $L4=y4(L2)$.

When you have collected all the data in L3 and L4, ask the calculator to multiply these corresponding slopes together and enter them in L5. Do this by moving to L5 and typing $L3*L4$.

What do you observe about the product of the slopes at the corresponding points?

Step 3:

Suppose you find the slope of a function f at (a,b) . How would you write this slope?

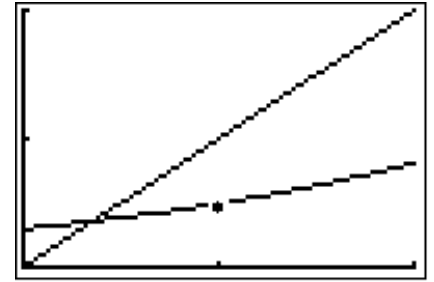
If the point (b,a) is the corresponding point on f^{-1} , how would you write the slope of f^{-1} at this point?

Write an relationship between the two slopes $f'(a)$ and $(f^{-1})'(b)$.

Additional problems to consider:

1. Suppose that f and f^{-1} are inverse functions that are differentiable and continuous on their domain. Suppose that $f(2)=3$ and $(f^{-1})'(3)=4$. Draw a sketch of two inverse functions that represent this situation. Find the slope of the function f at the point $(2,3)$.
2. Suppose that f and f^{-1} are inverse functions that are differentiable and continuous on their domain. Suppose that $(f^{-1})(2)=4$ and $(f^{-1})'(2)=1/2$. Draw a sketch of this function and a possible situation for the function f . What other information can you describe about the function f ?

3. Let f and f^{-1} be inverse functions that are differentiable and continuous on their domain. The graph of function f is shown at the right show the function f and a point $(1,0.5)$ along with the line $y=x$. Draw a sketch of f^{-1} and then find the corresponding point on f^{-1} . If the $(f^{-1})'(0.5) = 4$, find the slope of f at the point $(1, 0.5)$. Show the two tangent lines and label the slopes on both graphs.



4. Suppose that function $f(x) = 3 + 2x + e^x$. Find the derivative of f^{-1} at the point $x = 4$.

Charting Information About Slopes on Inverse Functions

Point on f		Corresponding point on f^{-1}		Slope of f at point	Slope of f^{-1} at corresponding point	Product of Slopes

INVESTIGATING LIMITS ON THE TI-84 CALCULATOR THROUGH TABLES

The expression $\lim_{x \rightarrow a} f(x) = L$ means that you can find a value L that the function $f(x)$ approaches as x approaches a . To examine this limit idea on the TI-84 graphing calculator you will use two features of the TI-84 calculator: the GRAPH and TABLE features.

Step 1: To investigate the $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$ enter the expression $\frac{1 - \cos x}{x}$ in Y_1 .

- GRAPH the function Y_1 in the standard window and look at the behavior of the function as x approaches 0.
- TRACE along this curve. What happens at $x = 0$? (If $x = 0$ is not a pixel value in this window redefine the side of the window by using Zoom-4: ZDecimal.)

Step 2: To observe more closely what this function is doing in a neighborhood of zero, use the TABLE feature. Touch 2ndTblSet. Set the TblMin = - 0.1 and the $\Delta Tbl = .01$.

- Observe what happens when the function approaches 0 at increments of .01.
- Reobserve the function from a TblMin = -0.01 and $\Delta Tbl = .001$. and once again from a TblMin = - 0.001 and $\Delta Tbl = .0001$. What is your

conclusion about $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$?

Exercises: Investigate the following limits using a graph and a table. Record at least 5 values of the function on either side of a . If you believe a limit exists then tell what that limit is. If a limit does not exist, describe why.

1. $\lim_{x \rightarrow 0} \frac{\sin(4x)}{4x}$

6. $\lim_{x \rightarrow 3} \frac{(x + 1)^2 - x^2}{1}$

2. $\lim_{x \rightarrow 0} \frac{|x|}{4x}$

7. $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$

3. $\lim_{x \rightarrow 3} \frac{(x + 1)^2 - x^2}{1}$

8. $\lim_{x \rightarrow 2} x^2 - \frac{2|x - 2|}{x - 2}$

4. $\lim_{x \rightarrow 0} \frac{\sin(x + 1) - \sin(x)}{1}$

9. $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{3x}$

5. $\lim_{x \rightarrow 0} \frac{\sin(3x)}{5x}$

10. $\lim_{x \rightarrow 0} \frac{|x + 3|}{2x + 7}$